Analysis of Rough Linear and Multilinear Pseudodifferential Operators

Manisha Malik

M.Sc. (H.S.) Mathematics, Panjab University
Email: malikmanisha1995@gmail.com

Abstract: **In the study of Partial Differential Equations and in Harmonic Analysis, an important role is played by the so-called *pseudodifferential operators*. For instance, for equations that describe electric potential and steady-state heat flow (elliptic equations) one can construct explicit solutions using pseudodifferential operators. Roughly speaking, these operators act on functions (or signals) by filtering (attenuating or amplifying) specific frequencies of those. For equations that describe wave propagation (hyperbolic equations), a similar role is played by *Fourier integral operators*. These tools allow us to obtain *a priori* estimates for the solutions, and study their behaviour and properties. Therefore, being able to estimate these operators in different function spaces is important for measuring the size and regularity of the solutions of PDEs in those spaces. We consider two types of multilinear pseudodifferential operators. First, we prove the boundedness of multilinear pseudo differential operators with symbols which are only measurable in the spatial variables in weighted Lebesgue spaces. These results generalise earlier work of the present authors concerning linear pseudo-pseudodifferential operators. Secondly, we investigate the boundedness of bilinear pseudodifferential operators with symbols in the H¨ormander S m ρ,δ classes. These results are new in the case ρ < 1, that is, outwith the scope of multilinear Calder´on-Zygmund theory.**

***Keywords*— Linear Pseudodifferential Operators, Multilinear Pseudodifferential Operators, analysis, Simulation**

1 Introduction

In mathematical analysis a pseudo-differential operator is an extension of the concept of differential operator. Pseudo-differential operators are used extensively in the theory of partial differential equations and quantum field theory.
Linear differential operators with constant coefficients
Consider a linear differential operator with constant coefficients,

$$P(D) := \sum_{\alpha} a_\alpha D^\alpha$$

Which acts on smooth functions

$$P(\xi) = \sum_{\alpha} a_\alpha \xi^\alpha,$$

With compact support in R^n. This operator can be written as a composition of a Fourier transform, a simple multiplication by the polynomial function (called the symbol)

$$P(D)u(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i(x-y)\xi} P($$

And an inverse Fourier transform, in the form:

$$D^\alpha = (-i\partial_1)^{\alpha_1} \cdots (-i\partial_n)^{\alpha_n}$$

2. Proof.
Case 1.

Suppose that the pair (f,k) satisfy the $CLCS$ property in $k(X)$. Then according to $CLCS$ property, there exists a sequence $< x_n >$ in X such that $\lim_{n\to\infty} fx_n = \lim_{x\to\infty} kx_n = t \in k(X)$ such that $t = kv$ for some $v \in X$. We claim that $gv = t$.
Putting $x = x_n, y = v$ in (2.1), we get

$$\int_0^{d(fx_n,gv)} \phi(t)dt \le \psi \int_0^{M(x_n,v)} \phi(t)dt$$

$$M(x_n, v) = \max\{d(kx_n, hv), d(kx_n, fx_n), d(gv, hv), \tfrac{1}{2}[d(kx_n, gv) + d(hv, fx_n)]\}$$

$$\lim_{n\to\infty} d(kx_n, hv) = 0$$

$$= \lim_{n\to\infty} d(kx_n, fx_n) = \lim_{n\to\infty} d(hv, f x_n)$$

$$\lim_{n\to\infty} d(f x_n, gv) = d(t, gv) = \lim_{n\to\infty} d(kx_n, gv)$$

$$= \lim_{n\to\infty} d(gv, hv)$$

Taking limit $n \to \infty$, we get

$$lim \int_0^{d(fx_n,gv)} \phi(t)dt \le \psi(\int_0^{d(t,gv)} \phi(t)dt$$

$< \int_0^{d(t,gv)} \phi(t)dt$, a contradiction.

$\therefore \lim_{n\to\infty} fx_n = \lim_{n\to\infty} kx_n = \lim_{n\to\infty} gv = \lim_{n\to\infty} hv = t$

Hence, v is a coincidence point of (f, k). Now, the weakly compatibility of pair (f,k) implies that $kfv = fkv = kt = ft$. We claim that t is a common fixed point of (f,k), that is ;$ft = kt = t$.

For putting $x = x_n, y = t$ in (2.1) and using $\lim_{n\to\infty} fx_n = t = \lim_{n\to\infty} kx_n,\ kt = gt,$ we have $\int_0^{d(fx_n,gt)} \phi(t)dt \le \psi\left(\int_0^{M(x_n,t)} \phi(t)dt\right)$

$$M(x_n,t) = max\left\{d(kx_n,ht), d(kx_n,fx_n), d(gt,ht), \frac{1}{2}[d(kx_n,gt) + d(ht,fx_n)]\right\}.$$

Taking limit as $n \to \infty$, we have

$$\lim_{n\to\infty} \int_0^{d(fx_n,gt)} \phi(t)dt \le \lim_{n\to\infty} \psi[\int_0^{M(x_n,t)} \phi(t)dt]$$

$$\lim_{n\to\infty} d(fx_n,gt) = \lim_{n\to\infty} d(kx_n,gt) = d(t,gt) = d(gt,ht)$$

$$\lim_{n\to\infty} d(kx_n,ht) = \lim_{n\to\infty} d(kx_n,fx_n) = \lim_{n\to\infty} d(ht,fx_n) = 0$$

$$\int_0^{d(t,gt)} \phi(t)dt \le \psi\left(\int_0^{d(t,gt)} \phi(t)dt\right)$$

If $d(t,gt) \ne 0$, then $d(t,gt) > 0$.

$$\int_0^{d(t,gt)} \phi(t)dt \le \psi\left(\int_0^{d(t,gt)} \phi(t)dt\right)$$

$< \int_0^{d(t,gt)} \phi(t)dt$, a contradiction.

$\therefore t = gt.$
Thus, $gt = t$. Here, in all cases, $gt = ht = t$.
It shows that $t \in k(X)$ is a common fixed point of (g,h).

Case 2.
Next, suppose that second pair (g,h) satisfy $CLCS$ property in a subcase $h(x)$. Then according to $CLCS$ property, ∃ a sequence $\{y_n\}$ in X such that $\lim_{n\to\infty} gy_n = \lim_{n\to\infty} hy_n \in h(X)$. So, ∃ $t' \in h(X)$ such that $t' = Su$ for some $u \in X$; where $t' = \lim_{n\to\infty} gy_n = \lim_{n\to\infty} hy_n$.
We claim that $fu = t'$ follows exactly as in case 1. It shows that u is coincidence point of (f,k). The weakly compatible of (f,k) implies that $fku = kfu = ft' = kt'$. It shows that t' is a coincidence point of (f,k) and $t \in h$(X).
Now, we claim that t' is a common fixed point of (f,k). This follows exactly as in Case 1, by putting $x = t', y = y_n$ in equation (2.1).
Taking $n \to \infty$ and using $ft' = kt'$.
Hence, $ft' = t'$.
It shows that $t' \in h(X)$ in common fixed point of (f,k).

Further, we claim that the common fixed point t' of (f,k) and t of (g,h) are same,that is $t = t'$. If not, then putting $x = t', y = t$ in condition (2), and using $ft' = kt' = t'$ and $gt = ht = t$,we have

$$\int_0^{d(ft',gt)} \phi(t)dt \le \psi(\int_0^{M(t,t')} \phi(t)dt$$

$$M(t',t) = max\begin{Bmatrix} d(kt',ht), d(kt',ft'), d(gt,ht), \\ 1/2\,[d(kt',gt) + d(ht,ft')] \end{Bmatrix}$$

$$\int_0^{d(t',t)} \phi(t)dt \le \psi(\int_0^{d(t',t)} \phi(t))dt)$$

$< \int_0^{d(t',t)} \phi(t)dt$, a contradiction.

Hence, $t = t'$.
This shows that t is a common fixed point of f,g,h,k.
Uniqueness of common fixed point follows easily. This completes the proof.

3 Applications

The study of multilinear pseudodifferential operators goes back to the pioneering works of R. Coifman and Y. Meyer, [6], [7], [8] and [9]. Since then, there has been a large amount of work on various generalisations of their results, as well as studies of bilinear operators with symbols satisfying different conditions to those in the standard bilinear Coifman-Meyer classes. The literature in this area of research is vast and any brief summary of it here would not do the authors justice. Therefore we confine ourselves to mention only those works with a direct connection to the present paper. R. Coifman and Y. Meyer, in [8] and [9], proved the boundedness from L p1 ×L p2 ×· · · ×L pN to L r of multilinear pseudodifferential operators with symbols in the class S 0 1,0 (n, N) (see Definition 5.3 below) for 1 < pi < ∞ and r > 1 with 1/p1 + 1/p2 + · · · + 1/pn = 1/r. In the seminal paper [11], L. Grafakos and R. Torres systematically developed the theory of multilinear Calder´onZygmund operators. They proved a multilinear T(1)-Theorem which they applied to generalise the result above to r > 1/N.
As a further application, they demonstrated the boundedness in Lebesgue spaces of multilinear pseudodifferential operators which, together with each of the adjoint operators, belonged to OP S0 1,1 (n, N) (see Definition 2.2). However, in [5], A. B´enyi and R. Torres showed that there exist symbols in S 0 1,1 (n, 2) that do not give rise to bilinear operators which are bounded from L p1 × L p2 to L r for 1 6 p1, p2, r < ∞ such that 1/p1 + 1/p2 = 1/r. In particular, there is no analogue of the Calder´on-Vaillancourt Theorem in the bilinear setting.
Moreover, the class of operators OP S0 1,1 (n, 2) is not closed under transposition. In contrast, [4] demonstrates that OP S0 1,0 (n, 2) is closed under transposition. Recently, in [2], A. B´enyi, D. Maldonado, V. Naibo and R. Torres proved that OP Sm ρ,δ(n, 2) is closed under transposition for 0 6 δ 6 ρ 6 1 and δ < 1. In particular, given an operator in OP S0 1,δ(n, 2), its adjoints are also in OP S0 1,δ(n, 2). Since S 0 1,δ(n, 2) ⊂ S 0 1,1 (n, 2), it follows that symbols

in S 0 1,δ(n, 2) give rise to bounded operators, by applying the result of [11] quoted above. In summary, we see that OP S0 ρ,δ(n, 2) are bounded on appropriate Lebesgue spaces when ρ = 1 (that is, the Calder´on-Zygmund case), but in general they fail to be bounded when ρ = 0. The purpose of this paper is to address the following question, which is of interest for ρ in-between these values, 'Given ρ 6 1, what m = m(ρ) 6 0 is sufficient to ensure that symbols in S m ρ,δ(n, N) give rise to bounded operators?'

This question is in the spirit of questions asked in [2]. We will study this question for two different symbol classes. First, in Section 3, we will consider a larger symbol class which does not require any differentiability in the spatial variable at all. That is, we study the multilinear symbol class L∞S m ρ (n, N) (see Definition 2.1) which, in particular, contains S m ρ,δ(n, N) for any δ. Our main result in this context is Theorem 3.3, which generalises a result obtained by the present authors in [14] regarding the linear case. The study of such symbol classes originates in [12], where C. Kenig and the third author studied linear operators. In the context of multilinear operators, results regarding mildly regular bilinear operators have been proved previously. In particular, D. Maldonado and V. Naibo established in [13] boundedness properties of bilinear pseudodifferential operators on products of weighted Lebesgue, Hardy, and amalgam spaces. The regularity they require in the spatial variables is only that of Dini-type. Section 4 deals with linear operators on mixed-norm Lebesgue spaces, and is a corollary to the proof of Theorem 3.3. The second topic we will study is the bilinear symbol class S m ρ,δ(n, 2). In Section 5 we adapt methods used to study symbols in L∞S m ρ (n, N) to weaken the requirement on m necessary to prove boundedness on Lebesgue spaces of operators in OP Sm ρ,δ(n, 2) for δ 6 ρ. This is formulated as Theorem 5.5. In Section 6, although we cannot show boundedness for general operators arising from symbols in S 0 ρ,δ(n, 2), we can prove boundedness on a suitable subclass. This is stated as Theorem 6.2, which is a result of the same flavour as that proved by F. Bernicot and S. Shrivastava in [3] regarding a subclass of OP S0 0,0 (1, 2), albeit proved by more straight-forward methods. A related result regarding OP S0 0,0 (n, 2) was also proved in [5].

4. Conclusion

In controlling height and width of a solution, the most important example of such spaces are the Lebesgue spaces Lp. Due to their rearrangement-invariant nature, these spaces are blind to the description of where solutions are concentrated, and thus the consideration of Lebesgue spaces with weights appears naturally. An important role is played by the so-called Muckenhoupt Ap weights.

For nonlinear PDEs , the multilinear counterpart of pseudodifferential and Fourier integral operators play a crucial role.

My recent research interests have been dealing with questions regarding both linear and multilinear operators of those described above, and in particular with those of rough type.

To get involved in a project in these areas requires a strong background and interest in Harmonic Analysis and PDEs. Some examples of lines of research that one could pursue:

- To develop an Ap-weighted theory for some classes of rough and mildly regular pseudodifferential operators, and find up-to-end-point improvements of existing results in the literature.
- To investigate the validity of corresponding end-point estimates for such operators.
- To develop the theory of spectral properties of rough pseudodifferential operators.
- Study multilinear end-point results and results of minimal regularity assumptions, for paraproducts and their application to the study of boundedness properties of multilinear pseudodifferential and Fourier integral operators.

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